DEMIMARTINGALES AND THE FUNCTIONAL HILL PROCESS FOR SMALL PARAMETERS

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Abstract. Association of random variables and Demimartingales are recent fields for handling asymptotic behaviors of sums of dependent random variables. We apply their techniques to establish the asymptotic law of the demimartingale

$$\sum_{j=1}^{k-1} f(j)(\exp(-\gamma \sum_{h=j+1}^{k-1} E_h/h) - \exp(-\gamma \sum_{h=j}^{k-1} E_h/h)),$$

where $E_1, E_2, ...$ are independant standard exponential random variables, $\gamma > 0$, k is a positive integer and $f(j)$ is an increasing function of the integer $j \geq 1$. We next apply the results to find the asymptotic behavior the functional Hill process for small parameters whithin the Extreme Value Theory (EVT) field. Such a result would have been very hard to find whithout demimartingales techniques.

1. Introduction

We are concerned in this paper with some demimartingale and its application in extreme value theory. Namely let $n \geq 1$, $1 < k(n) < n$, $\gamma > 0$, let $E_{1,n}, E_{2,n}, ..., E_{n,n}$ be a sequence of independent standard exponential random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $f(j)$ be a nondecreasing function of $j \in \mathbb{N}\backslash\{0\}$, and finally let

$$W_{k(n),n}(f) = \sum_{j=1}^{k(n)-1} f(j)(\exp(F_{j+1,n}) - \exp(F_{j,n})), \tag{1.1}$$

where

$$F_{j,n} = -\gamma \sum_{h=j}^{k-1} E_{h,n}/h, \;\; F_{k,n} = 0.$$

In this note, we intend to use the relatively recent theory of associated random variables and demimartingales to determine the limiting law

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$W_{k(n),n}(f)$ when

$$\sum_{j=1}^{k-1}(f(j)-f(j-1))j^{-1/2}<\infty, \tag{K1}$$

and next to apply the results to solve an open problem in statistics of the extreme value Theory. We then may begin to introduce to associated random variables which goes back to Lehmann (1966) [11] in the bivariate case.

The concept of associated variable generalizes that of independence and seems to model a great variety of stochastic models. *We point out now that we will equally say that a sequence of random variables is associated or that the elements of the sequence are associated.* As immediate examples, Gaussian random vectors with nonnegatively correlated components(see Pitt [12]). Also the order statistics associated with a finte set of independent are associated . As well a homogenuous Markov chain is also associated (Daley [2]). Such a property also arises in Physics, and is quoted under the name of FKG property (Fortuin, Kastelyn et Ginibre (1971) [7]), in percolation theory and even in Finance (see Pan Jiazhu [16]). The final definition is given by Esary, Proschan et Walkup (1967) [6] in

Definition 1. *A finite sequence of rv's $(X_1,...,X_n)$ are associated when for couple of real and coordinate-wise nondecreasing functions h et g defined on $\mathbb{R}^n$, we have*

$$Cov(h(X_1,...,X_n),\quad g(X_1,...,X_n))\geq 0 \tag{1.2}$$

An infinite sequence of rv's are associated whenever all its finite subsequences are associated.

We have a few number of interesting properties to be found in Rao ([13]) : **(P1)** A sequence of independent rv's is associated. (P2) Partial sums of associated rv's are associated. **(P3)** Nondecreasing functions of associated variables are associated. **(P4)** Let the sequence $Z_1,Z_2,...,Z_n$ be associated and let $(a_i)_{1\leq i\leq n}$ be positive numbers and $(b_i)_{1\leq i\leq n}$ real numbers. Then the rv's $a_i(Z_i-b_i)$ are associated.

Demimartingales are set from associated centered variables exactly as martingales are derived from partial sums of centered independent random variables. We have

Definition 2. *A sequence of rv's $\{S_n, n \geq 1\}$ in $L^1(\Omega, F, P)$ is a demimartingale when for any $j \geq 1$,for any coordinatewise nondecreasing function g defined on $\mathbb{R}^d$, we have*

$$\mathbb{E}\big((S_{j+1} - S_j)\ g(S_1, ..., S_j)\big) \geq 0, \quad j \geq 1. \tag{1.3}$$

Two particular cases should be pointed out. First any martingale is a demimartingale. Secondly, partial sums $S_n = X_1 + ... + X_n$ of associated and centred randoms variables $X_1, X_2, ...$ form a demimartingale for, in this case, (1.3) becomes :

$$\mathbb{E}\left\{(S_{j+1} - S_j)\ g(S_1, ..., S_j)\right\} = \mathbb{E}\left\{X_{j+1}\ g(S_1, ..., S_j)\right\} = Cov\left\{X_{j+1}\ g(S_1, ..., S_j)\right\},$$

since $EX_{j+1} = 0$. Since $(x_1, ..., x_{j+1}) \longmapsto x_{j+1}$ et $(x_1, ..., x_{j+1}) \longmapsto g(x_1, ..., x_j)$ are coordinate-wise nondecreasing functions and since the $X_1, X_2, ..$ are associated, we get

$$\mathbb{E}\left\{(S_{j+1} - S_j)\ g(S_1, ..., S_j)\right\} = Cov\left\{X_{j+1}\ g(S_1, ..., S_j)\right\} \geq 0.$$

We claim that this thoery is enough to handle the limiting law issue of $W_{k,n}$ and its application in statistics of the extreme to solve open problems. We organize the remainder of the paper as follows. In Subsection 2.1 of Section 2, we link $W_{k,n}$ to demimartingales theory and provide our main result on $W_{k,n}$ with the help of recent theorems on demimartingales in Subsection 2.2. In Section 3, we give some applications on the functional Hill process and concluding remarks. The paper is finished by an appendix about moment computations.

2. Main results and proofs

2.1. **How demimartingales may help?**

2.1.1. *Preliminary lemma.*

Lemma 1. *For each $n > 1$, the family $\{E(W_{k,n}) - W_{k,n}, 1 < k < n\}$ is a demimartingale.*

2.1.2. *Proof.* We have

$$W_{k,n} = f(k-1) - \left(\sum_{j=1}^{j=k-1} (f(j) - f(j-1)) \exp(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h)\right)$$

$$= f(k-1) - \sum_{j=1}^{j=k-1} \overline{f}(j) S^*_{j,k,n}$$

with $S^*_{j,k,n} = \exp(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h)$, $\overline{f}(j) = f(j) - f(j-1)$, $f(0) = 0$.

We have by (4.1.1)

$$s_{j,k}^* = E(S_{j,k,n}^*) = \exp(-\sum_{h=j}^{k-1} \log(1+\gamma/h))$$

and

$$E(W_{k,n}) = f(k-1) - \sum_{j=1}^{j=k-1} \overline{f}(j) s_{j,k,n}^* \tag{2.1}$$

and next

$$W_{k,n}^* = E(W_{k,n}) - W_{k,n} = \sum_{j=1}^{j=k-1} \overline{f}(j)(S_j^* - s_j^*) \tag{2.2}$$

which is a sum of centred associated rv's. By the properties (P1)-(P6) above, the $\overline{f}(j)(S_j^* - s_j^*)$ are associated since $\overline{f}(j) > 0$ for all $j > 1$. Then their partial sums form a demimartingale.

2.2. **The main result.**

Theorem 1. *Let $f(j)$ be an increasing function of the integer $j \geq 1$ such that*

$$\sum_{j=1}^{\infty} f(j) j^{-1/2} < \infty. \tag{2.3}$$

and let $1 \leq k \leq n$. Then $W_{k,n}(f)$ converges p.s to a finite random variable $W(f)$ such that $E|W(f)| < \infty$. Further if $f(j) = j^{\tau}, 0 < \tau < 1/2$, then

$$D_{k,n}(\tau) = \sum_{j=1}^{j=k-1} ((j+1)^{\tau} - j^{\tau})(S_j^* - s_j^*)$$

also converges to a random variable $W(\tau)$ such that $E|W(\tau)| < \infty$.

2.3. **Proof of Theorem 1.**

We begin to consider an infinite sequence of independent and standard exponential random variables $E_1, E_2, ...$ defined on the same probability space. Under the hypotheses of the theorem, we have

$$W_k^*(f) = \sum_{j=1}^{k-1} f(j)(\exp(-\gamma \sum_{h=j+1}^{k-1} E_h/h) - \exp(-\gamma \sum_{h=j}^{k-1} E_h/h)), k \geq 1,$$

with the convention $\sum_{h=k}^{k-1} E_h/h = 0$. We have for any any function f, $1 < k < n$,

$$W_k^*(f) =_d W_{k,n}^*(f),$$

where $=_d$ stands for the equality in distribution. It is clear that if we prove that $W_k^*(f)$ converges *a.s.* to a finite random variable $W(f)$ as $k \to \infty$, we will able to conclude that $W_{k,n}(f)$ converges in distribution to $W(f)$ as $k = k(n) \to +\infty$. Let us now prove that $W_k^*(f)$ converges *a.s.* to a finite random variable $W(f)$ as $k \to \infty$ by applying Theorm 2.4.2 of Rao ([13]) for demimartingales since $W_k^*(f)$ also forms a demimartingale. Is suffices to check that

$$\limsup_{k\to\infty} E\left|W_k^*\right| < \infty \tag{2.4}$$

Denote $S_{j,k}^* = \exp(-\gamma \sum_{h=j}^{k-1} E_h/h), 1 \leq j \leq k-1$ and $S_{k,k}^* = 1$. By Cauchy-Schwarz's inequality and next by Minkoswki's one, we have

$$E\left|W_k^*\right| \leq (EW_k^{*2})^{1/2} = \|W_k^*\|_2 \leq \sum_{j=1}^{k-1} \left\|\overline{f}(j)(S_{j,k}^* - s_{j,k}^*)\right\|_2$$

$$\leq \sum_{j=1}^{k-1} \overline{f}(j)(Var(S_{j,k}^*))^{1/2}.$$

Now by (4.2), $\overline{f}(j)(Var(S_{j;k}^*))^{1/2}$ is dominated by $Const.\ \overline{f}(j)j^{-1/2}$ for large values of j. Thus Rao's condition (2.4)is valid and this gives the first part of the proof. Now for $f(j) = j^\tau$,

$$\overline{f}(j)j^{-1/2} \sim Const.\ j^{-((3/2)-\tau)}$$

and $\sum j^{-((3/2)-\tau)}$ converges if and only if $\tau < 1/2$. This gives the second part of the Theorem 1.

2.4. **Finite-dimensional limiting law.** Consider the space $\mathcal{F}$ of functions increasing functions $f(j)$ of $j \geq 1$ satisfying (2.4) and consider $\{f_1, ..., f_r\} \subset F, r \geq 2$. We surely have

Theorem 2. *Let $r \geq 2$ and $\{f_1, ..., f_r\} \subset F$ and let $a_1, ..., a_r$. Then $\sum_{s=1}^r a_r W_{k,n}^*(f_s)$ converges a.s. to a random variable $\sum_{s=1}^r a_r W^{(s)}(f_s)$ where each has the same law as $W(f_s)$ found in Theorem..*

This directly results from Theorem 1 and does not need to be proved.

3. Application to Extreme value Theory

3.1. **Asymptotic results in the Weibull case.** The following stochastic process

$$T_n(f) = \sum_{j=1}^{j=k(n)} f(j) \left(\log X_{n-j+1,n} - \log X_{n-j,n}\right) \tag{3.1}$$

was introduced by Deme et al.(2012) [3] as a generalization of the Diop and Lo continuous generalization of the Hill statistic for $f(j) = j^{\tau}, \tau > 1$. The Hill statistic corresponding to $\tau = 1$ and will be denoted here by $T_n(1)$. This latter plays a key role in Univariate Extreme Value Theorem (UEVT).

This theory has its foundations in finding the asymptotic law of the maximum observation $X_{n,n} = \max(X_1, ..., X_n)$. It is said that the underlying distribution function F of the observations is attracted to some df H if for some sequences $(a_n > 0)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, we have for any continuity point of H,

$$\lim_{n\to\infty} P(\frac{X_{n,n} - b_n}{a_n} \leq x) = \lim_{n\to\infty} F^n(a_n x + b_n) = H(x).$$

It is known that, when it is nondegenerated, H can be parametrized as $G_{\gamma}(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), 1 + \gamma x > 0$, $\gamma \in \mathbb{R}$ named as the Generalized Pareto Distribution (GDP). It is said that F is in the domain of attraction of G_{γ}, hereby denoted as : $F \in D(G_{\gamma})$. The reader is referred to de Haan and Feirreira [9] , Resnick [14], Galambos [8] and Beirlant *et al.* [1] for a modern account of UEVT.

Although the parameter γ of the GDP is continuous, the three cases $(\gamma < 0)$, $\gamma = 0$ and $\gamma > 0$, respectively denamed Weibull, Gumbel and Frechet cases, may behave ridically differently. But in all the cases, the Hill statistic is used to estimate what is called the extremal index in the following sense : For $\gamma \geq 0$, $k^{-1}T_n(1) \to \gamma$ as $n \to +\infty$ and $k/n \to 0$; for $\gamma < 0$, then the upper enpoint of $G(x) = F(e^x)$ defined by $y_0 = \log \sup\{x \in R, F(x) < 1\}$, is finite and $k^{-1}T_n(1) / (x_0 - G^{-1}(1-k/n)) \to (1-\gamma)^{-1}$ $n \to +\infty$ and $k/n \to 0$ and G^{-1} stands for the generalized inverse function of G.

The Diop and Lo generalization of Hill

$$D_n(\tau) = \sum_{j=1}^{j=k(n)} j^{\tau} \left(\log X_{n-j+1,n} - \log X_{n-j,n}\right), \tau > 0,$$

has been studied in ([5]) where its asymptotic normality was proved for any γ but for $\tau > 1/2$. Recently, the functional form $T_n(f)$, which generalizes $D_n(\tau) = T_n(f_{\tau}) = D_n(\tau)$, has been extendely studied in the Frechet and Gumbel cases by Deme *et al.*([3]) who proved this : $T_n(f)$ has a Gaussian limiting process when $A(2, f) = \sum_{j=1}^{k} f(j)^2/j^2 = +\infty$ and

$$B_n(f) = \max\{f(j)^2/j^2, 1 \leq j \leq k\}/(\sum_{j=1}^{k} f(j)^2/j^2)^{1/2}$$

$\rightarrow 0$ as $n \rightarrow \infty$. It has a non Gaussian limiting process when $A(2, f) < +\infty$.

When particularized for f_τ, we get that asymptotic normality holds for $\tau \geq 1/2$ and not for $0 < \tau < 1/2$. Their results are based on sums of independent random variables, and then on Kolmogorov type theorems (see [10]). When put together, for the class of functions f_τ, we remark that the beahavior of $T_n(f_\tau)$ is known for any γ in the whole extremal domain except for the Weibull domain and for $0 < \tau < 1/2$, that if for small parameters $\tau' s$.

This problem remained unsolved, may be by the fact that its depends of sums of dependent data and that we didnot have the appropriate frame. Its is clear know that the demimartingale may help to solve it. In this paper, let consider the very simple case of

$$x_0 - G^{-1}(1-u) = u^{1/\gamma}, 0 \leq u \leq 1.$$

We use here the index $-\gamma < 0$ intead of $\gamma < 0$. We remark that $G(x) = F(e^x) \in D(G_{-1/\gamma})$ if and only if $F \in D(G_{-1/\gamma})$. We use the classical representation of the $Y_j = \log X_j$ associated with the distribution function $G(x) = F(e^x)$ through a sequence of independant standard uniform random variables $U_1, U_2; ...$, that is

$$\{Y_j, j \geq 1\} =_d \{G^{-1}(1-U_j), j \geq 1\}$$

and then

$$\{\{Y_{n-j+1,,n}, 1 \leq j \leq n\}, n \geq 1\} =_d \left\{\{G^{-1}(1-U_{j,n}), 1 \leq j \leq n\}, n \geq 1\right\}.$$

This gives

$$T_n(f)/(y_0 - Y_{n-k+1,n}) = \sum_{j=1}^{j=k(n)} f(j) \left(\log X_{n-j+1,n} - \log X_{n-j,n}\right)$$

$$= \sum_{j=1}^{j=k(n)} f(j) \frac{((y_0 - \log X_{n-j,n}) - (y_0 - \log X_{n-j+1,n}))}{(y_0 - Y_{n-k+1,n})}$$

$$=_d \sum_{j=1}^{j=k(n)} f(j) U_{k,n}((U_{j+1,n}/U_{k,n})^\gamma - (U_{j,n}/U_{k,n})^\gamma).$$

We have for $1 \leq j \leq k-1$,

$$(U_{j,n}/U_{k,n})^\gamma = \prod_{h=j}^{k-1} (U_{h,n}/U_{h+1,n})^\gamma = \exp(-\gamma \sum_{h=j}^{k-1} \frac{1}{h} \log(U_{h+1,n}/U_{h,n})^h)$$

$$\equiv \exp(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h).$$

By the Malmquist representation (see ([15]), p. 336), the random variables $E_h^{(n)}, 1 \leq h \leq n$, are independent and standard ones. We arrive at

$$T_n(f)/(y_0 - Y_{n-k+1,n})$$

$$= \sum_{j=1}^{k(n)} f(j) \left\{ \exp(-\gamma \sum_{h=j+2}^{k-1} E_h^{(n)}/h) - \exp(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h) \right\}$$

and by (2.1),

$$D_{k,n}(f) - \left(T_n(f)/(y_0 - Y_{n-k+1,n}) \right) = W_{k,n}^*(f),$$

where $D_{k,n} = f(k-1) - \sum_{j=1}^{k-1} \overline{f}(j) s_{j,k}^*$. At this step, we apply Theorem 1 to get the final result

Proposition 1. *Let $X_1, X_2, ...$ be positive random variable with a finite endpoint x_0 such that $\log F^{-1}(1-u) = \log x_0 + u^\gamma, 0 \leq u \leq 1$ and $\gamma > 0$. Let f be an increasing function over the positive integers satisfying $(K1)$ and put*

$$D_{k,n}(f) = f(k-1) - \sum_{j=1}^{k-1} \overline{f}(j) s_{j,k}^*$$

Then

$$T_n^*(f) = D_{k,n} - \left(T_n(f)/(\log x_0 - \log X_{n-k+1,n}) \right)$$

converges in distribution to a finite random variable $W(f)$ with finite expectation. As well, for $0 < \tau < 1/2, T_n^*(f_\tau)$ converges in distribution to a finite random variable $W(f_\tau)$ with finite expectation.

We indeed remark that for this simple case in the Weibull case, the law of the functional Hill process is determined for $0 < \tau < 1/2$. For the general case, we have the following Karamata representation when F is in the Weibull case of parameter $\gamma > 0 : x_0(F) < \infty$ and

$$\log x_0 - F^{-1}(1-u) = (1 + p(u)) u^\gamma \exp(\int_u^1 b(t) t^{-1} dt), \tag{3.2}$$

where $(p(u), b(u)) \to (0, 0)$ as $u \to 0$. In a coming paper, we will determine general conditions on f, b and p under which $T_n^*(f)$ bahaves as $W_{k,n}^*$ as in the present case.

3.2. **Critical points of the distribution function of $W(f)$.** We use computer-based methods for approximating the law of $W(f)$. Simulation studies show that the empirical distribution functions based of $B0 = 1000$ replications are very stable from $k = 2000$.

We proceed as follows. Fix $\tau, 0 < \tau < 1/2$, $\gamma > 0$ and $k \geq 2000$. At each step B from 1 to $B0 = 1000$, we generate $E_1(B), ..., E_k(B)$ and compute W_k^* denoted by $W_k^*(B)$. We finally consider the empirical *df*, denoted by G_k, based on $W_k^*(1), ..., W_k^*(B0)$. Since G_k is stable in the sense that it does not significally change from $k = 2000$, we do approximate the *df* G_∞ of $W(f_\tau)$ by G_k for k large enough.

As an example, we illustrate in Figure 1 the *df* G_k for $k = 250$, 500, 750, 1000, 2000, 500 for $\gamma = 1$ and $\tau = 1/4$. Here for instance, we fer that the support of G_∞ is $[-0.5, 0.5]$. On the whole, the figures cleary establish stability and support our proposal. For users interested to use our method, we provide an executable file located at :

http://www.ufrsat.org/lerstad/resources/lmhfw1.exe

for the computation of $P(W(f_\tau) \leq x) = G_\infty(x)$ and $P(|W(f_\tau)| \leq |x|) = G_\infty(|x|) - G_\infty(-|x|)$ for $x \in \mathbb{R}$.

3.3. **Statistical tests.** Let us illustrate here how G_∞ may be used to test the hypothesis that $F \in D(G_{-1/\gamma})$. We use here the following approximation :

$$T_n^*(f) \approx T_n(f)/(\log X_{n,n} - \log X_{n-k+1,n}).$$

We consider gere the statistical test **(H)** : $F \in D(G_{-1/\gamma})$ and compute the p-values for the models as precised in Table 1. The first three *df*'s are in the Weibull domain with $\gamma = 1$. The first (Weibull 1) is the one we used in the paper. In the two others (Weibull 2 and Weibull 3), we introduce a shift of order $(1-u)^5$. For these cases, we conclude that the hypothesis is confirmed. But further computations show that for $n = 300$ and $k = 200$, the hypothesis is rejected for a shift of order $(1-u)^q$ with $0 \leq q \leq 3$. This is conceivable since, as we pointed out above, the convergence depends on the functions b and p in (3.2) that are here $b(u) = 0$ and $p(u) = u^q$. This dependence of the results on the auxilliary functions will be studied in a coming paper.

As for the two last cases, the $p - values$ is zero and the related models are rejected as expected since the Exponential rv is in the Gumbel domain and the Pareto one in the Frechet one.

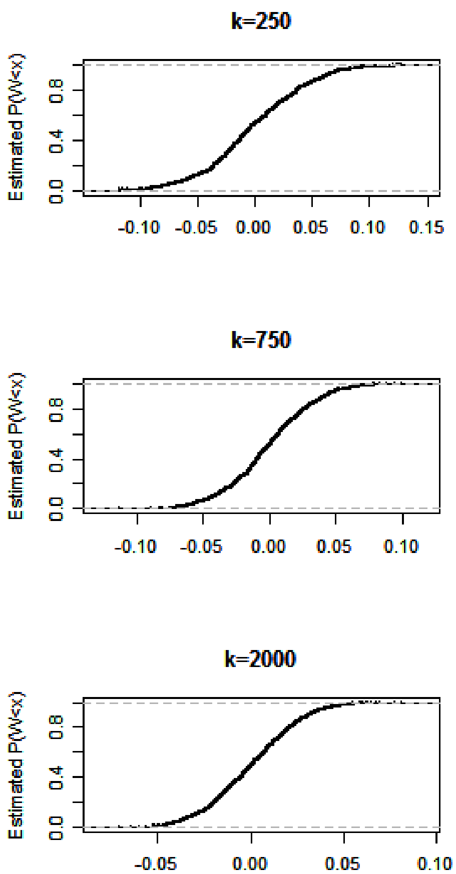

FIGURE 1. Illustration the distribution functions of $W_{k,n}(1/4)$ for different values of k

Models	Quantiles functions	$T_n^*(f_\tau)$	P-values
Weibull 1	$F^{-1}(u) = \exp(1 + u^\gamma)$	-0.01897	50.6%
Weibull 1	$F^{-1}(u) = \exp(1 + u^\gamma(1 + (1-u)^5))$	-0.02524	39.6%
Weibull 2	$F^{-1}(u) = \exp(1 + u^\gamma(1 + (1-u)^5))$	-0.012556	66.6%
Standard Expnential	$F^{-1}(u) = -\log(1-u)$	0.5313	0%
Pareto	$F^{-1}(u) = 1 + u^{-1}$	0.9969	0%

TABLE 1. Statistical tests for four models using the law of $W(1/4)$

4. Appendix

This section is devoted to the computations of the moments of

$$S_j^* = exp(-\gamma \sum_{h=j}^{k-1} E_h/h)$$

where $E_h's$ are independent standard exponential random variables, and their approximations for large values of j. We begin to give a particlular and useful for the the expansion of the logarithm function.
Fact 1. Let $\varepsilon > 0$ be fixed for once. There exists $0 < u_0$ such that

$$0 < u < u_0, \ \log(1+u) = u + \theta(\varepsilon, u)u^2,$$

where $\theta(\varepsilon, u) \in [-\varepsilon - 1/2, \varepsilon - 1/2] \equiv A(\varepsilon) = [a_1(\varepsilon), a_2(\varepsilon)]$. For any integer $m \geq 1$, let $J_0(m)$ such that $J_0(m) \geq \gamma/(mu_0)$ so that

$$j \geq J_0(m) \Longrightarrow \log(1+\gamma/j) = u + \theta_j u^2 \text{ with } \theta_j \in A(\varepsilon).$$

In the remainder, we concentrate on the moment computations.

4.1. Moment estimation.

4.1.1. *Exact values.* We have for any integer $m \geq 1$,

$$\mathbb{E}\left((S_j^*)^m\right) = E(\exp(-m\gamma \sum_{h=j}^{k-1} E_h/h) = \prod_{h=j}^{k-1} E(-m\gamma E_h/h) = \prod_{h=j}^{k-1}(1+m\gamma/h)^{-1}$$

$$\mathbb{E}\left((S_j^*)^m\right) = \exp\left(-\sum_{h=j}^{k-1}\log(1+m\gamma/h)\right).$$

Now for $j \geq J_0(m)$,

$$\mathbb{E}\left((S_j^*)^m\right) = \exp\left(-m\gamma\sum_{h=j}^{k-1}(1/h) - m^2\gamma^2\sum_{h=j}^{k-1}\theta_h/h^2\right).$$

4.1.2. *Approximated values for moments.* We have by

$$\left|m^2\gamma^2\sum_{h=j}^{k-1}\theta_h/h^2\right| \leq |a_1(\varepsilon)|\, m^2\gamma(\frac{1}{j} - \frac{1}{k-1} - \frac{1}{(k-1)^2}) \leq \frac{|a_1(\varepsilon)|\, m^2\gamma}{j}.$$

For

$$\frac{|a_1(\varepsilon)|\, m^2\gamma}{J_1(\varepsilon, m)} \leq \varepsilon,$$

we have

$$j \geq J_1(\varepsilon, m) \vee J_0(m) \Longrightarrow \exp\left(-m^2\gamma^2\sum_{h=j}^{k-1}\theta_h/h^2\right) \leq e^{\varepsilon}.$$

Next by (4.4),

$$\exp(-m\gamma\sum_{h=j}^{k-1}(1/h)) = \left(\frac{j}{k-1}\right)^{m\gamma}\exp(-m\gamma\left\{\sum_{h=j}^{k-1}\frac{1}{h}-\log((k-1)/j)\right\})$$

with

$$\exp(-1/j)\leq\exp(-m\gamma\left\{\sum_{h=j}^{k-1}\frac{1}{h}-\log((k-1)/j)\right\})\leq\exp(-1/(k-1)).$$

We finally have for $j\geq J_1(\varepsilon,m)\vee J_0(m)$,

$$\mathbb{E}\left((S_j^*)^m\right)=\left(\frac{j}{k-1}\right)^{m\gamma}B(1,m,j)B(2,m,j), \tag{4.1}$$

with

$$0\leq B(1,j)=1+O(\frac{|a_1(\varepsilon)|\,m^2\gamma}{j})\text{ and }B(2,j)=1+O(j^{-1})).$$

4.1.3. *Approximated values for variances.* We have for $j>J_0(2)$

$$\mathbb{E}\left((S_j^*)^2\right)=\exp\left(-2\gamma\sum_{h=j}^{k-1}(1/h)-4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2\right).$$

and for $j>J_0(1)$

$$\mathbb{E}\left(S_j^{*2}\right)=\left(-\gamma\sum_{h=j}^{k-1}(1/h)-\gamma^2\sum_{h=j}^{k-1}\theta_h(2)/h^2\right)^2$$

$$=\exp\left(-2\gamma\sum_{h=j}^{k-1}(1/h)-2\gamma^2\sum_{h=j}^{k-1}\theta_h(2)/h^2\right).$$

Thus

$$Var(S_j^*)=\exp(2\gamma\sum_{h=j}^{k-1}1/h)$$

$$\times\left\{\exp(-4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2)-\exp(-2\gamma^2\sum_{h=j}^{k-1}(2\theta_h(1)-\theta_h(2))/h^2)\right\}.$$

Since where $x=4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2$ and $y=2\gamma^2\sum_{h=j}^{k-1}\theta_h(2)/h^2$ are both nonnegative, we have $|e^x-e^y|\leq|x-y|$. Thus

$$0\leq\exp(-4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2)-\exp(-2\gamma^2\sum_{h=j}^{k-1}\theta_h(2))/h^2)$$

$$\leq 2\gamma^2 \sum_{h=j}^{k-1} |2\theta_h(1) - \theta_h(2)| /h^2 \leq \frac{2\gamma^2 |a_1(\varepsilon)|}{j},$$

by (4.5). Hence

$$Var(S_j^*) = \left(\frac{j}{k-1}\right)^{2\gamma} V(1,j)V(2,j) \tag{4.2}$$

with

$$|V(1,j)| = 1 + O(j^{-1}) \text{ and } 0 \leq V(2,j) \leq \frac{2\gamma^2 |a_1(\varepsilon)|}{j}.$$

4.1.4. *Covariance approximate values.* Let $\ell > 1$ and consider $\sigma_{j,j+\ell} = cov(S_{j+\ell}^*, S_j^*)$. We have

$$E\left(S_j^*\right) = \exp(\sum_{h=j}^{k-1} - \log(1+\gamma/h))$$

$$= \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h)) \exp(\sum_{h=j+\ell}^{k-1} - \log(1+\gamma/h))$$

$$= E\left(S_{j+\ell}^*\right) \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h)).$$

Also

$$S_j^* S_{j+\ell}^* = \exp(-\gamma \sum_{h=j}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j+\ell}^{k-1} E_h/h)$$

$$= \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h - \gamma \sum_{h=j+\ell}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j+\ell}^{k-1} E_h/h)$$

$$= \exp(-2\gamma \sum_{h=j+\ell}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h) = \left(S_{j+\ell}^*\right)^2 \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h).$$

Hence

$$E(S_j^* S_{j+\ell}^*) = E\left(S_{j+\ell}^*\right)^2 \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h)).$$

For $j \geq J_0(1) \vee J_0(2)$,

$$cov(S_j^*, S_{j+\ell}^*) = Var(S_{j+\ell}^*) \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h))$$

$$cov(S_j^*, S_{j+\ell}^*) = Var(S_{j+\ell}^*) \exp(-\gamma \sum_{h=j}^{j+\ell-1} 1/h - \gamma^2 \sum_{h=j}^{j+\ell-1} \theta_h/h^2)$$

$$= Var(S_{j+\ell}^*) \left(\frac{j}{j+\ell-1} \right)^{\gamma} (1+O(j^{-1}))(1+) \tag{4.3}$$

4.2. **Integral computations.** Let $b \geq 1$, we get by comparing the area under the curve $x \longmapsto x^{-b}$ from j to $k-1$ and those of the rectangles based on the intervals $[h, h+1]$, $j = 1, .., k-2$, we get

$$\sum_{h=j}^{k-2} h^{-b} \leq \int_j^{k-1} x^{-b} dx \leq \sum_{h=2}^{k-1} h^{-b},$$

that is

$$\int_j^{k-1} x^{-b} dx - j^{-b} \leq \sum_{h=j}^{k-1} h^{-b} \leq \int_j^{k-1} x^{-b} dx - (k-1)^{-b}.$$

For $b = 1$, on has

$$\frac{1}{k-1} \leq \log((k-1)/j) - (\sum_{h=j}^{k-1} \frac{1}{h}) \leq \frac{1}{j}. \tag{4.4}$$

For $b = 2$, on has

$$\frac{1}{j} - \frac{1}{k-1} - \frac{1}{j^2} \leq \sum_{h=j}^{k-1} h^{-2} \leq \frac{1}{j} - \frac{1}{k-1} - \frac{1}{(k-1)^2}, \tag{4.5}$$

that is

$$\frac{1}{(k-1)^2} \leq \frac{1}{j}(1 - \frac{j}{k-1}) - \sum_{h=j}^{k-1} h^{-2} \leq \frac{1}{j^2}.$$

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